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Tensor products of Cohen-Macaulay rings

Solution to a problem of Grothendieck

S. BOUCHIBA
Department of Mathematics
University Moulay Ismail
Meknes 50000, Morocco
Email address: bouchiba@fsmek.ac.ma

S. KABBAJ
Department of Mathematics
Harvard University
Cambridge, MA 02138, USA
Email address: kabbaj@math.harvard.edu

ABSTRACT

In this paper we solve a problem, originally raised by Grothendieck, on the transfer of Cohen-Macaulayness to tensor products of algebras over a field k. As a prelude to this, we investigate the grade for some specific types of ideals that play a primordial role within the ideal structure of such constructions.

INTRODUCTION

All rings and algebras considered in this paper are commutative with identity elements and, unless otherwise specified, are to be assumed to be non-trivial. All ring homomorphisms are unital. Throughout, k stands for a field. Let A be a ring. We shall use $G(I)$ to denote the grade of an ideal I of A, $Z(A)$ to denote the set of all zero-divisors of A, and $k_A(p)$ to denote the quotient field of $\frac{A}{p}$ for any prime ideal p of A.

Let A be a Noetherian ring and I a proper ideal of A. The grade of I is defined to be the common length of all maximal A-sequences in I. It can be measured by the (non-) vanishing of certain Ext modules. In fact, according to [10, Theorem 16.7], $G(I) = \mathrm{Inf}\{i | \mathrm{Ext}^i_A(\frac{A}{I}, A) \neq 0\}$. This connection opened commutative algebra to the application of homological methods. Finally, recall that the Cohen-Macaulay rings are those Noetherian rings in which grade and height coincide for every ideal.

Our aim in this paper is to prove that the Cohen-Macaulay property is inherited by tensor products of k-algebras. To this purpose, the first section investigates the grade of three specific types of ideals that play a primordial role within the ideal structure of the tensor product of two k-algebras. This allows us, in the second section, to establish the main theorem, that is, for k-algebras A and B such that $A \otimes_k B$ is Noetherian, $A \otimes_k B$ is a Cohen-Macaulay ring if and only if so are A and B.

Suitable background on depth of modules and Cohen-Macaulay rings is [5], [8], [9], and [10]. For a geometric treatment of the Cohen-Macaulay property, we refer the reader to the excellent book of Eisenbud [6]. Recent developments on heights of primes and dimension theory in tensor products of k-algebras are to be found in [1], [2], and [13]. Any unreferenced material is standard, as in [7], and [9].

1. GRADE OF IDEALS IN A TENSOR PRODUCT OF TWO k-ALGEBRAS

The grade of an arbitrary ideal in a (Noetherian) tensor product of two k-algebras seems to be difficult to grasp. It would appeal to new techniques yet to be discovered. Our goal here is much more modest. We shall determine the grade of three specific types of ideals that play a primordial role within the ideal structure of this construction.

We announce the main result of this section.

Theorem 1.1. Let A and B be k-algebras such that $A \otimes_k B$ is Noetherian. Let I and J be proper ideals of A and B, respectively. Then:

a) $G(I \otimes_k B) = G(I)$ and, similarly, $G(A \otimes_k J) = G(J)$.

b) $G(I \otimes_k B + A \otimes_k J) = G(I) + G(J)$.

c) $G(I \otimes_k J) = \mathrm{Inf}(G(I), G(J))$.

Let A and B be two k-algebras. Let x be a non zero-divisor element of A and y a non zero-divisor element of B. Then $x \otimes y$ is a non zero-divisor element of $A \otimes_k B$. Let I be a proper ideal of A. Then $I \otimes_k B$ is a proper ideal of $A \otimes_k B$. If $x_1, ..., x_n$ is an A-sequence, then it is easily seen that $x_1 \otimes 1, ..., x_n \otimes 1$ is an $(A \otimes_k B)$-sequence. These basic facts will be used frequently in the sequel without explicit mention. Moreover, we assume familiarity with the natural isomorphisms for tensor products, as in [3]. In particular, we identify A and B with their respective images in $A \otimes_k B$, and if I and J are proper ideals of A and B, respectively, then $\frac{A \otimes_k B}{I \otimes_k B + A \otimes_k J} \cong \frac{A}{I} \otimes_k \frac{B}{J}$. Also, we recall that $A \otimes_k B$ is a free (hence flat) extension of A and B.

The proof of the main theorem requires the following preparatory lemma.

Recall first that if A is a ring and $x_1, ..., x_n$ are elements of A, then $x_1, ..., x_n$ is said to be a permutable A-sequence if any permutation of the x's is also an A-sequence.

Lemma 1.2. Let A and B be two k-algebras. Let $x_1, ..., x_n$ be a permutable A-sequence and $y_1, ..., y_n$ be a permutable B-sequence. Then $x_1 \otimes y_1, ..., x_n \otimes y_n$ is a permutable $(A \otimes_k B)$-sequence.

Proof. The argument follows easily from the combination of the next two statements. The proofs of these are straightforward and hence left to the reader.

1) If $x_1, x_2, ..., x_n$ is a permutable A-sequence, then so is $x_1x_2, x_3, ..., x_n$.

2) If $x_1, ..., x_n$ is a permutable A-sequence and $y_1, ..., y_m$ is a permutable B-sequence, then $x_1 \otimes 1, ..., x_n \otimes 1, 1 \otimes y_1, ..., 1 \otimes y_m$ is a permutable $(A \otimes_k B)$-sequence. □

Proof of the theorem. a) Let $G(I) = n$ and $x_1, ..., x_n$ be an A-sequence in I. Then $x_1, ..., x_n$ is an $A \otimes_k B$-sequence in $I \otimes_k B$. Since $I \subseteq Z(\frac{A}{(x_1,...,x_n)})$, there exists $a \in A \setminus (x_1, ..., x_n)$ such that $Ia \subseteq (x_1, ..., x_n)$ [9, Theorem 82]. Then

$$\begin{aligned}(I \otimes_k B)a &= Ia \otimes_k B \\ &\subseteq (x_1, ..., x_n) \otimes_k B \\ &= (x_1, ..., x_n).\end{aligned}$$

Clearly, $a \notin (x_1, ..., x_n) \otimes_k B$. Hence $I \otimes_k B \subseteq Z(\frac{A \otimes_k B}{(x_1,...,x_n)})$. Consequently, $G(I \otimes_k B) = n = G(I)$. Likewise for $G(A \otimes_k J) = G(J)$.

b) Let $G(I) = n$ and $G(J) = m$. Let $x_1, ..., x_n$ be an A-sequence in I and $y_1, ..., y_m$ a B-sequence in J. Obviously, $x_1, ..., x_n, y_1, ..., y_m$ is an $A \otimes_k B$-sequence in $I \otimes_k B + A \otimes_k J$. Since $I \subseteq Z(\frac{A}{(x_1,...,x_n)})$ and $J \subseteq Z(\frac{B}{(y_1,...,y_m)})$, there exist $a \in A \setminus (x_1, ..., x_n)$ and $b \in B \setminus (y_1, ..., y_m)$ such that $Ia \subseteq (x_1, ..., x_n)$ and $Jb \subseteq (y_1, ..., y_m)$. Then

$$\begin{aligned}(I \otimes_k B + A \otimes_k J)(a \otimes b) &\subseteq Ia \otimes_k B + A \otimes_k Jb \\ &\subseteq (x_1, ..., x_n) \otimes_k B + A \otimes_k (y_1, ..., y_m) \\ &= (x_1, ..., x_n, y_1, ..., y_m).\end{aligned}$$

Since $\overline{a} \neq \overline{0}$ in $\frac{A}{(x_1,...,x_n)}$ and $\overline{b} \neq \overline{0}$ in $\frac{B}{(y_1,...,y_m)}$, then $\overline{a} \otimes \overline{b} \neq \overline{0}$ in $\frac{A \otimes_k B}{(x_1,...,x_n,y_1,...,y_m)}$, whence $(I \otimes_k B + A \otimes_k J) \subseteq Z(\frac{A \otimes_k B}{(x_1,...,x_n,y_1,...,y_m)})$. Consequently, $G(I \otimes_k B + A \otimes_k J) = G(I) + G(J)$, as asserted.

c) Let $G(I) = n \leq G(J) = m$. By [9, Exercise 23, p. 104], there exist a permutable A-sequence $x_1, ..., x_n$ in I and a permutable B-sequence $y_1, ..., y_m$ in J. By Lemma 1.2, $x_1 \otimes y_1, ..., x_n \otimes y_n$ is an $A \otimes_k B$-sequence in $I \otimes_k J$. Since, by (a), $n = G(I \otimes_k B) \geq G(I \otimes_k J)$, it follows that $G(I \otimes_k J) = n$, as desired. □

2. WHEN IS THE TENSOR PRODUCT OF TWO k-ALGEBRAS A COHEN-MACAULAY RING?

Recall that a Cohen-Macaulay ring is a Noetherian ring A in which $G(M) = ht(M)$ for every maximal ideal M of A [9, Definition, p. 95]. It is worthwhile noting, according to [9, Theorem 136], that grade and height coincide for every proper ideal in a Cohen-Macaulay ring. In general, the inequality height $\geq$ grade holds. We next show that, in $A \otimes_k B$, the assumption of equality of grade and height for ideals of the form $p \otimes_k B + A \otimes_k q$, where p and q are prime ideals of A and B, respectively, implies equality for all ideals.

In 1965, Grothendieck proved in [8, (6.7.1.1)] that if K and L are extension fields of k one of which is finitely generated over k, then $K \otimes_k L$ is a Cohen-Macaulay ring. In 1969, Watanabe et al. extended this result showing, in [14, Theorem], that if A and B are Cohen-Macaulay rings such that $A \otimes_k B$ is Noetherian and $\frac{A}{m}$ is a finitely generated field extension of k for each maximal ideal m of A, then $A \otimes_k B$ is a Cohen-Macaulay ring.

Our purpose in this section is to prove the following:

Theorem 2.1. Let A and B be k-algebras such that $A \otimes_k B$ is Noetherian. Then the following statements are equivalent:

i) $A \otimes_k B$ is a Cohen-Macaulay ring ;

ii) $G(I \otimes_k B + A \otimes_k J) = ht(I \otimes_k B + A \otimes_k J)$, for all proper ideals I and J of A and B, respectively ;

iii) $G(p \otimes_k B + A \otimes_k q) = ht(p \otimes_k B + A \otimes_k q)$, for all prime ideals p and q of A and B, respectively ;

iv) A and B are Cohen-Macaulay rings.

The discussion which follows, concerning basic facts about k-algebras, will provide some background to the main theorem and will be of use in its proof. We shall use $t.d.(A : k)$ to denote the transcendence degree of a k-algebra A over k. It is worth reminding the reader that for an arbitrary k-algebra A (not necessarily a domain), $t.d.(A : k) := \text{Sup}\{t.d.(\frac{A}{p} : k) | p \in \text{Spec}(A)\}$ (cf. [13, p. 392]).

Notice first that the tensor product of two extension fields of k is not necessarily Noetherian [12]. However, given two k-algebras A and B such that $A \otimes_k B$ is Noetherian, then A and B are necessarily Noetherian rings; moreover, either $t.d.(A : k) < \infty$ or $t.d.(B : k) < \infty$: indeed, since A and B each have only a finite number of minimal prime ideals, there exist $p \in \text{Spec}(A)$ and $q \in \text{Spec}(B)$ such that $t.d.(A : k) = t.d.(\frac{A}{p} : k)$ and $t.d.(B : k) = t.d.(\frac{B}{q} : k)$. Clearly, $k_A(p) \otimes_k k_B(q)$ is Noetherian, since it is a localization of $\frac{A}{p} \otimes_k \frac{B}{q} \cong \frac{A \otimes_k B}{p \otimes_k B + A \otimes_k q}$, which is Noetherian. We obtain, by [12, Corollary 10], that either $t.d.(k_A(p) : k) < \infty$ or $t.d.(k_B(q) : k) < \infty$, as desired.

The proof of the main theorem requires two preparatory results.

Lemma 2.2. Let K and L be two extension fields of k such that $K \otimes_k L$ is Noetherian. Then $K \otimes_k L$ is a Cohen-Macaulay ring.

Proof. Step 1. We claim that $K \otimes_k A$ is a Cohen-Macaulay ring provided K is an algebraic extension field of k and A is a Cohen-Macaulay ring such that $K \otimes_k A$ is Noetherian. Indeed, let $P \in \mathrm{Spec}(K \otimes_k A)$ and $p = P \cap A$. Since $K \otimes_k A$ is a flat integral extension of A, $ht(P) = ht(p)$ [13, Lemma 2.1]. By Theorem 1.1, $G(p) = G(K \otimes_k p) \leq G(P)$. Therefore $ht(P) = ht(p) = G(p) \leq G(P) \leq ht(P)$ [9, Theorem 138]. Thus $G(P) = ht(P)$. Consequently, $K \otimes_k A$ is a Cohen-Macaulay ring.

Step 2. Let K and L be any extension fields of k such that $K \otimes_k L$ is Noetherian. We may suppose that $t = t.d.(K : k) < \infty$. Let $x_1, ..., x_t$ be elements of K algebraically independent over k. Then $K \otimes_k L \cong K \otimes_{k(x_1,...,x_t)} S^{-1}L[x_1, ..., x_t]$ ([11, Proposition 2.6]), where $S = k[x_1, ..., x_t] \setminus \{0\}$. Since K is an algebraic extension field of $k(x_1, ..., x_t)$ and $A = S^{-1}L[x_1, ..., x_t]$ is a Cohen-Macaulay ring ([9, Theorem 151 and Theorem 139]), by Step1, $K \otimes_k L \cong K \otimes_{k(x_1,...,x_t)} A$ is a Cohen-Macaulay ring. □

Proposition 2.3. Let A and B be k-algebras such that $A \otimes_k B$ is Noetherian. Let P be a prime ideal of $A \otimes_k B$, $p = P \cap A$, and $q = P \cap B$. Then

a) $ht(P) = ht(p) + ht(q) + ht(\frac{P}{p \otimes_k B + A \otimes_k q})$.

b) $G(P(A \otimes_k B)_P) = G(pA_p) + G(qB_q) + ht(\frac{P}{p \otimes_k B + A \otimes_k q})$.

Proof. a) Consider the canonical flat homomorphism of Noetherian rings

$$f : A \to A \otimes_k B.$$

Applying [10, Theorem 15.1], we have

$$\begin{aligned} ht(P) &= ht(p) + dim(\tfrac{(A \otimes_k B)_P}{p(A \otimes_k B)_P}) \\ &= ht(p) + dim(\tfrac{(A \otimes_k B)_P}{(p \otimes_k B)_P}) \\ &= ht(p) + dim((\tfrac{A \otimes_k B}{p \otimes_k B})_{\frac{P}{p \otimes_k B}}) \\ &= ht(p) + ht(\tfrac{P}{p \otimes_k B}). \end{aligned}$$

Similarly, via the canonical homomorphism of Noetherian rings

$$g : B \to \frac{A}{p} \otimes_k B,$$

we get

$$\begin{aligned} ht(\tfrac{P}{p \otimes_k B}) &= ht(q) + ht(\tfrac{P/(p \otimes_k B)}{\frac{A}{p} \otimes_k q}) \\ &= ht(q) + ht(\tfrac{P/(p \otimes_k B)}{(p \otimes_k B + A \otimes_k q)/(p \otimes_k B)}) \\ &= h(q) + ht(\tfrac{P}{p \otimes_k B + A \otimes_k q}). \end{aligned}$$

It follows that $ht(P) = ht(p) + ht(q) + ht(\frac{P}{p \otimes_k B + A \otimes_k q})$, as desired.

b) Notice first that $k_A(p) \otimes_k k_B(q)$ is a Cohen-Macaulay ring, by Lemma 2.2. Set $S_1 = A \setminus p$, $S_2 = B \setminus q$, and $S = \{a \otimes b | a \in S_1 \text{ and } b \in S_2\}$.

The above homomorphism f induces the local flat homomorphism of Noetherian rings $A_p \to (A \otimes_k B)_P$. In view of [10, Corollary, p. 181] or [8, Proposition IV-6.3.1], we have

$$\begin{aligned} G(P(A \otimes_k B)_P) &= \text{depth}((A \otimes_k B)_P) \\ &= \text{depth}(A_p) + \text{ depth}\left(\frac{(A\otimes_k B)_P}{pA_p(A\otimes_k B)_P}\right) \\ &= G(pA_p) + \text{ depth}\left(\frac{(A\otimes_k B)_P}{(p\otimes_k B)_P}\right) \\ &= G(pA_p) + \text{ depth}\left(\left(\frac{A}{p} \otimes_k B\right)_{\frac{P}{p\otimes_k B}}\right). \end{aligned}$$

In a similar way, via the induced local flat homomorphism $B_q \to (\frac{A}{p} \otimes_k B)_{\frac{P}{p\otimes_k B}}$ of g, we get

$$\text{depth}\left(\left(\frac{A}{p} \otimes_k B\right)_{\frac{P}{p\otimes_k B}}\right) = G(qB_q) + \text{ depth}\left(\left(\frac{A}{p} \otimes_k \frac{B}{q}\right)_{\frac{P}{p\otimes_k B + A\otimes_k q}}\right).$$

It follows that

$$\begin{aligned} G(P(A \otimes_k B)_P) &= G(pA_p) + G(qB_q) + \text{ depth}\left(\left(\frac{A}{p} \otimes_k \frac{B}{q}\right)_{\frac{P}{p\otimes_k B + A\otimes_k q}}\right) \\ &= G(pA_p) + G(qB_q) + \text{ depth}((k_A(p) \otimes_k k_B(q))_H) \\ &= G(pA_p) + G(qB_q) + \dim((k_A(p) \otimes_k k_B(q))_H), \end{aligned}$$

where $H = \frac{S^{-1}P}{S_1^{-1}p\otimes_k S_2^{-1}B + S_1^{-1}A\otimes_k S_2^{-1}q}$. Consequently, $G(P(A \otimes_k B)_P) = G(pA_p) + G(qB_q) + ht(\frac{P}{p\otimes_k B + A\otimes_k q})$, as desired. □

Proof of the theorem. i)⇒ ii) and ii)⇒ iii) are obvious. Assume that (iii) holds. Let $p \in \text{Spec}(A)$ and $q \in \text{Spec}(B)$. Then, by Theorem 1.1, $G(p \otimes_k B + A \otimes_k q) = G(p) + G(q)$. On the other hand, by Proposition 2.3, $ht(p \otimes_k B + A \otimes_k q) = ht(p) + ht(q)$. Hence, since $G(p \otimes_k B + A \otimes_k q) = ht(p \otimes_k B + A \otimes_k q)$, $G(p) + G(q) = ht(p) + ht(q)$. Therefore $ht(p) - G(p) = G(q) - ht(q)$, so that $G(p) = ht(p)$ and $G(q) = ht(q)$, making (iv) hold.

Now, suppose that (iv) holds. By [9, Theorem 140], it is sufficient to prove that $(A\otimes_k B)_P$ is a Cohen-Macaulay ring for each prime ideal P of $A\otimes_k B$. Let P be a prime ideal of $A\otimes_k B$, $p = P \cap A$, and $q = P \cap B$. By Proposition 2.3, $G(P(A \otimes_k B)_P) = G(pA_p) + G(qB_q) + ht(\frac{P}{p\otimes_k B + A\otimes_k q})$ and $ht(P(A \otimes_k B)_P) = ht(P) = ht(p) + ht(q) + ht(\frac{P}{p\otimes_k B + A\otimes_k q})$. Since A and B are Cohen-Macaulay, A_p and B_q are Cohen-Macaulay. Then $G(pA_p) = ht(p)$ and $G(qB_q) = ht(q)$. Therefore $G(P(A \otimes_k B)_P) = ht(P(A \otimes_k B)_P)$. Then $(A \otimes_k B)_P$ is a Cohen-Macaulay ring. Hence (i) holds. □

Remark 2.4. One may prove directly (i)⇔(iv) of Theorem 2.1 by using Lemma 2.2 and [8, Corollaire IV-6.3.3]. However, our proof is designed to draw extra benefits: Proposition 2.3 and Theorem 2.1(ii) & (iii) shed more light on the prime ideal structure of (Noetherian) tensor products of k-algebras. Further, in the Noetherian case, Proposition 2.3(a) stands for a satisfactory analogue of [4, Theorem 1], a central result for polynomial rings.

Remark 2.5. Theorem 2.1 may allow one to determine the grade for new categories of primes of $A \otimes_k B$ (different from those treated in Theorem 1.1). For instance, let $P \in$

Spec$(A \otimes_k B)$ with $p = P \cap A$ and $q = P \cap B$. Assume that p and q are generated by an A-sequence and a B-sequence, respectively. Then

$$G(P) = G(p) + G(q) + G(\frac{P}{p \otimes_k B + A \otimes_k q}).$$

If, in addition, p and q are maximal ideals, then

$$G(P) = G(p) + G(q) + ht(\frac{P}{p \otimes_k B + A \otimes_k q}).$$

Indeed, let $p = (x_1, ..., x_n)$ and $q = (y_1, ..., y_m)$ such that $x_1, ..., x_n$ is an A-sequence and $y_1, ..., y_m$ is a B-sequence. Clearly, $x_1, ..., x_n, y_1, ..., y_m$ is an $A \otimes_k B$-sequence in $p \otimes_k B + A \otimes_k q$ with $p \otimes_k B + A \otimes_k q = (x_1, ..., x_n, y_1, ..., y_m)$. By [9, Theorem 116], $G(\frac{P}{p \otimes_k B + A \otimes_k q}) = G(P) - (n + m) = G(P) - G(p) - G(q)$.

Assume now that p and q are maximal ideals. Then, applying Theorem 2.1, $\frac{A \otimes_k B}{p \otimes_k B + A \otimes_k q} \cong \frac{A}{p} \otimes_k \frac{B}{q}$ is a Cohen-Macaulay ring. It follows that $G(\frac{P}{p \otimes_k B + A \otimes_k q}) = ht(\frac{P}{p \otimes_k B + A \otimes_k q})$, as desired.

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